

The Classification of the Simply Laced Berger Graphs from Calabi-Yau CY_3 spaces

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ABSTRACT: The algebraic approach to the construction of the reflexive polyhedra that yield Calabi-Yau spaces in three or more complex dimensions with K3 fibres reveals graphs that include and generalize the Dynkin diagrams associated with gauge symmetries. In this work we continue to study the structure of graphs obtained from CY_3 reflexive polyhedra. The objective is to describe the “simply laced” cases, those graphs obtained from three dimensional spaces with K3 fibers which lead to symmetric matrices. We study both the affine and, derived from them, non-affine cases. We present root and weight structurea for them. We study in particular those graphs leading to generalizations of the exceptional simply laced cases $E_{6,7,8}$ and $E_{6,7,8}^{(1)}$. We show how these integral matrices can be assigned: they may be obtained by relaxing the restrictions on the individual entries of the generalized Cartan matrices associated with the Dynkin diagrams that characterize Cartan-Lie and affine Kac-Moody algebras. These graphs keep, however, the affine structure present in Kac-Moody Dynkin diagrams. We conjecture that these generalized simply laced graphs and associated link matrices may characterize generalizations of Cartan-Lie and affine Kac-Moody algebras.

KEYWORDS: .

1. Introduction

Progress in fundamental physics is dependent on the identification of underlying symmetries such as general coordinate invariance or gauge invariance. The final objective of this work is to look for possible symmetries beyond those of the Standard Model. The latter is based on Cartan-Lie Algebras and their direct products, and is very successful. There have been valiant efforts to extend the Standard Model within the framework of Cartan-Lie algebras and with the objective of, for example, reducing the number of free parameters appearing in the theory. However, attempts to formulate Grand Unified theories in which the direct product of the symmetries of the Standard Model is embedded in some larger simple Cartan-Lie group have not had the same degree of success as the Standard Model. The alternative possibility of unifying the gauge interactions with gravity in some ‘Theory of Everything’ based on string theory is very enticing, in particular because this offers novel algebraic structures.

At a very basic level, and without any obvious direct interest for the content of the Standard Model, Cartan-Lie symmetries are closely connected to the geometry of symmetric homogeneous spaces, which were classified by Cartan himself. Subsequently, an alternative geometry of non-symmetric spaces appeared, and their classification was suggested in 1955 by Berger using holonomy theory [1]. There are several infinite series of spaces with holonomy groups $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$ and $Sp(n) \times Sp(1)$, and additionally some exceptional spaces with holonomy groups $G(2)$, $Spin(7)$, $Spin(16)$.

Superstring theories offer new clues how to attack the problem of the nature of symmetries at a very basic geometric level. For example, the compactification of the heterotic string leads to the classification of states in a representation of the Kac-Moody algebra of the gauge group $E_8 \times E_8$ or $Spin(32)/Z_2$. These structures arose in compactifications of the heterotic superstring on 6-dimensional Calabi-Yau spaces, non-symmetric spaces with an $SU(3)$ holonomy group [2]. It has been shown [3] that group theory and algebraic structures play basic roles in the generic two-dimensional conformal field theories (CFTs) that underlie string theory. The basic ingredients here are the central extensions of infinite-dimensional Kac-Moody algebras. There is a clear connection between these algebraic and geometric generalizations. Affine Kac-Moody algebras are realized as the central extensions of loop algebras, namely the sets of mappings on a compact manifold such as S^1 that take values on a finite-dimensional Lie algebra. Superstring theory contains a number of other infinite-dimensional algebraic symmetries such as the Virasoro algebra associated with conformal invariance and generalizations of Kac-Moody algebras themselves, such as hyperbolic and Borcherd algebras.

In connection with Calabi-Yau spaces, (Coxeter-)Dynkin diagrams which are in one-to-one correspondence with both Cartan-Lie and Kac-Moody algebras have been revealed through the technique of the crepant resolution of specific quotient singular structures such as the Kleinian-Du-Val singularities \mathbf{C}^2/G [22], where G is a discrete subgroup of $SU(2)$. Thus, the rich singularity structure of some examples of non-symmetrical Calabi-Yau spaces provides another opportunity to uncover infinite-dimensional affine Kac-Moody symmetries. The Cartan matrices of affine Kac-Moody groups are identified with the

intersection matrices of the unions of the complex projective lines resulting from the blow-ups of the singularities. For example, the crepant resolution of the \mathbf{C}^2/Z_n singularity gives for rational, i.e., genus-zero, (-2) curves an intersection matrix that coincides with the A_{n-1} Cartan matrix. This is also the case of $K3 \equiv CY_2$ spaces, where the classification of the degenerations of their elliptic fibers (which can be written in Weierstrass form) and their associated singularities leads to a link between CY_2 spaces and the infinite and exceptional series of affine Kac-Moody algebras, $A_r^{(1)}$, $D_r^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$ (ADE) [23, 24].

The study of the Calabi-Yau spaces appearing in superstring, F and M theories can be approached via the theory of toric geometry and the Batyrev construction [4] using reflexive polyhedra. The concept of reflexivity or mirror symmetry has been linked [5] to the problem of the duality between superstring theories compactified on different $K3$ and CY_3 spaces. The same Batyrev construction has also been used to show how subsets of points in these reflexive polyhedra can be identified with the Dynkin diagrams [5–8] of the affine versions of the gauge groups appearing in superstring and F-Theory. More explicitly, the gauge content of the compactified theory can be read off from the *dual* reflexive polyhedron of the Calabi-Yau space which is used for the compactification.

In the case of a $K3 = CY_2$ Calabi-Yau space, any subdivision of the reflexive polyhedron into different subsets separated by a polygon which is itself reflexive is equivalent to establishing a fibration structure for the space, whose fiber is simply being the space corresponding to the intermediate mirror polygon. For example, a reflexive polyhedron intersected by a plane yields a planar reflexive polygon separating the ‘top’ and ‘bottom’ subsets in the nomenclature of [5], called ‘left’ and ‘right’ in this work. Subsets of points in these reflexive polyhedra are those which can be identified with the Dynkin diagrams [5–8] of the affine Kac-Moody algebras. It is however necessary to stress that this task was facilitated by the a priori knowledge of the fiber structure, the reflexive Weierstrass triangle in those cases. Until the emergence of the UCYA, the absence of a systematic way of determining the slice structure in generic CY_n has prevented further progress in this area and the finding of new Dynkin or generalized Dynkin diagrams.

Since Calabi-Yau spaces may be characterized geometrically by reflexive Newton polyhedra, they can be enumerated systematically [25]. Moreover, one can beyond simple enumeration, as it has been recently realized that different reflexive polyhedra are related algebraically via what has been termed the Universal Calabi-Yau Algebra (UCYA). The term ‘Universal’ is motivated by the fact that it includes ternary and higher-order operations, as well as familiar, beyond binary operations.

The UCYA is particularly well suited for exploring the fibrations of Calabi-Yau spaces, which are visible as lower-dimensional slice or projection structures in the original polyhedra. The knowledge of the slice structure (see table (1) in this work of some illustration) allows us to uncover and understand not only Dynkin structures in $K3$ and elliptic polyhedra but new graphs in CY_n polyhedra. For an example of an elliptic fibration of a $K3$ space, see Fig. (1) in Ref.[9] and its accompanying description. The ‘left’ and ‘right’ parts of this reflexive polyhedron both correspond to so-called ‘extended vectors’. In the UCYA scheme, the binary operation of summing these two extended vectors gives a true reflexive vector, that characterizes the full $CY_2 = K3$ manifold. establishing a direct algebraic

relation between $K3(= CY_2)$ and CY_3 spaces. This property is completely general: it has been shown previously how the UCYA, with its rich structure of binary and higher-order operations, can be used to generate and interrelate CY_n spaces of any order. The UCYA provides a complete and systematic description of the analogous decompositions or nestings of fibrations in Calabi-Yau spaces of any dimension.

One of the remarkable features of Fig. (1) in Ref.[9] is that the right and left sets of nodes constitute graphs corresponding to affine Dynkin diagrams: namely the $E_6^{(1)}$ and $E_8^{(1)}$ diagrams. This is not a mere coincidence or an isolated example. As discussed there, in Ref.[9], all the elliptic fibrations of $K3$ spaces found using the UCYA construction feature this decomposition into a pair of graphs that can be interpreted as Dynkin diagrams.

The purpose of this paper is to continue the work already initiated in Refs.[9, 16] on the generalization of the previous results for $K3$ spaces to Calabi-Yau spaces in any dimension and with any fiber structure. The main objective here is to describe the “simply laced” cases, those graphs obtained from three dimensional spaces with $K3$ fibers which lead to symmetric matrices. As was first shown in Ref.[9], many new diagrams - which we term ‘Berger Graphs’ - can be found in this way. In Ref.[16] we gave a more formal and comprehensive definition of Berger graphs and matrices. Some examples of planar and non-planar diagrams obtained from CY_3 were presented and studied. It was seen there how some of those diagrams could be extended into infinite series while some others could be considered exceptional, not extendable. We hypothesize that Berger graphs correspond, in some manner that remains to be defined, to some new algebraic structure, just as Dynkin diagrams are in one-to-one correspondence with root systems and Cartan matrices in semi-simple Lie Algebras and affine Kac-Moody algebras.

Our final objective would be to construct a theory similar to Kac-Moody algebras, in which newly extensions of Cartan matrices fulfilling generalized conditions are introduced. There are plenty of possible generalizations of Cartan matrices obtainable by modifying the rules for the diagonal and off-diagonal entries in the matrices, and it is impossible to find all of them and classify them. On the other hand, probably not all of them give meaningful, consistent generalizations of Kac-Moody algebras, and probably fewer of them have interesting implications for physics. One has to find natural conditions on these matrices, hopefully inspired by physics. The relation of Berger Graphs to Calabi-Yau spaces could be this inspirational physical link. Once one has the equivalent of the Cartan matrix, one can use standard algebraic tools, such as the definition of an inner product, the construction of a root system, its group of transformations, etc., which could be helpful in clarifying the meaning and significance of this construction.

The structure of this paper is as follows. In section 2 we show how to extract graphs directly from the polyhedra associated with Calabi-Yau spaces and how one can define new, related graphs by adding or removing nodes. We make the important remark that this is possible in the UCYA formalism because because of its ability to give naturally the complex slice structure of the Calabi-Yau spaces. In section 3 we present a review of the formal algebraic definition of Berger graphs and matrices. In section 4 we present a list of simply laced graphs obtained from CY_3 spaces and give a general description of their properties. On continuation we illustrate these properties in some more detail with some

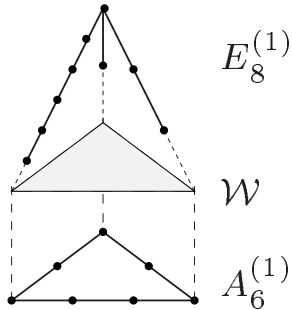


Figure 1: The schematic decomposition of the the K3 polyhedron determined by $\vec{k}_4 = (1, 1, 2, 3)$ [7] with an elliptic Weierstrass intersection/projection (\mathcal{W}) gives the two $A_6^{(1)}$ and $E_8^{(1)}$ Dynkin diagrams.

examples. Finally, in section 4 we draw some conclusions and make some conjectures.

2. UCYA and generalized Dynkin diagrams.

One of the main results in the Universal Calabi-Yau Algebra (UCYA) is that the reflexive weight vectors (RWVs) \vec{k}_n of dimension n , which are the fundament for the construction of CY spaces, can obtained directly from lower-dimensional RWVs $\vec{k}_1, \dots, \vec{k}_{n-r+1}$ by algebraic constructions of arity r [10–13]. The dimension of the corresponding vector is $d + 2$ for a Calabi-Yau CY_d space.

For example, the sum of vectors, a binary composition rule of the UCYA, gives complete information about the $(d - 1)$ -dimensional slice structure of CY_d spaces. In the K3 case, the Weierstrass fibered 91 reflexive weight vectors of the total of 95 \vec{k}_4 can be obtained by such binary, or arity-2, constructions out of just five RWVs of dimensions 1,2 and 3.

In an iterative process, we can combine by the same 2-ary operation the five vectors of dimension 1,2,3 with these other 95 vectors to obtain a set of 4242 chains of five-dimensional RWVs \vec{k}_5 CY_3 chains. This process is summarized in Fig. (3) in Ref.[9]. By construction, the corresponding mirror CY_3 spaces are shown to possess K3 fiber bundles. In this case, reflexive 4-dimensional polyhedra are also separated into three parts: a reflexive 3-dimensional intersection polyhedron and ‘left’ and ‘right’ skeleton graphs. The complete description of a Calabi-Yau space with all its non-trivial d_i fiber structures needs a full range of n-ary operations where $n_{max} = d + 2$.

It has been shown in the toric-geometry approach how the Dynkin diagrams of affine Cartan-Lie algebras appear in reflexive K3 polyhedra [4–8]. We present an illustratory example in Fig.(1) where the decomposition of a K3 polyhedron with an elliptic Weierstrass intersection gives as a result two Dynkin diagrams for $A_6^{(1)}$ and $E(8)^{(1)}$. This example is not an isolated one, all the elliptic fibrations of K3 spaces found using the UCYA technique feature this kind of decomposition into a pair of graphs that can be interpreted as Dynkin diagrams.

As it has been pointed out in the introduction and as it is obvious from the figure(1) the task of discerning Dynkin diagrams among all the set of points was facilitated by the a priori knowledge of the intersecting polyhedra, the Weierstrass triangle in that cases. UCYA gives naturally the slice structure in the reflexive polyhedra and the projective structure in the corresponding mirror polyhedra. The knowledge of these slices is a necessary first step in the uncovering and understanding of new Dynkin or generalized Dynkin diagrams (our Berger diagrams).

It has also been shown [10, 12, 13], using examples of the lattice structure of reflexive polyhedra for $CY_n : n \geq 2$ with elliptic fibers, that there is a correspondence between the five basic RWVs (basic constituents of composite RWVs describing K3 spaces, see section 2 in Ref.[9]) and affine Dynkin diagrams for the five ADE types of Lie algebras (A, D series and exceptional $E_{6,7,8}$).

In each case, a pair of extended RWVs have an intersection which is a reflexive plane polyhedron; each vector from the pair gives the left or right part of the three-dimensional RVW. The construction generalizes to any dimension. In Ref.[9] it was remarked that in the corresponding “left” and right “graphs” of $CY_{3,4,..}$ Newton reflexive polyhedra one can find new graphs with some regularity in its structure.

In principle one should be able to build, classify and understand these regularities of the graphs according to the n-arity operation which originated the construction. For the case of binary or arity-2 constructions: two graphs are possible. In general for any reflexive polyhedron, for a given arity-r intersection, it corresponds exactly r graphs.

In the binary case, the 2-ary intersection (a plane) in the Newton polyhedra, which correspond to the *eldest* reflexive vector of the series, separate left and right graphs. A concrete rule for the extraction of individual graph points from all possible nodes in the graphs is that they are selected if they exactly belong “on the edges” lying on one side or another with respect the intersection, see figure(1). In the ternary case, the 3-ary intersection hypersurface is a volume, which separate three domains in the newton polyhedra and three graphs are possible. Individual points are assigned to each graph looking at their position with respect to the volume intersection (see Tab.(1) in Ref.[9]) for some aclaratory examples).

The emergence of Dynkin diagrams or generalized Berger diagrams in Calabi-Yau reflexive polyhedra is not a mere philatelic curiosity: in a concrete singular limit of the K3 space, there appears a gauge symmetry whose Cartan-Lie algebra corresponds to the Dynkin diagram seen as a graph on one side of Fig. (1). In general, the rich singularity structures of $K3 \equiv CY_2$ spaces are closely connected to the affine Cartan-Lie symmetries $A_r^{(1)}$, $D_{2r}^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$ via the crepant resolution of specific quotient singular structures such as the Kleinian-Du-Val singularities \mathbf{C}^2/G [22], where G is a discrete subgroup of $SU(2)$. For example, the crepant resolution of the \mathbf{C}^2/Z_n singularity gives for rational, i.e., genus-zero, (-2) curves an intersection matrix that coincides with the A_{n-1} Cartan matrix. Also, in the case of K3 spaces with elliptic fibers which can be written in Weierstrass form, there exists an ADE classification of degenerations of the fibers [23, 24].

Graphs can directly be obtained from the reflexive polyhedron construction but can also be defined graphs independently of it. New graphs will be derived, or by direct ma-

nipulation of the original ones, or from generalized Cartan matrices in a purely algebraic fashion. They will basically consist on the primitive graphs extracted from reflexive polyhedra to whom internal nodes in the edges will have been added or eliminated. The nature of the relation, if any, of the graphs thus generated to the geometry of Toric varieties and the description of Calabi-Yau as hypersurfaces on them is related to the possibility of defining viable “fan” lattices. This is an open question, clearly related to the properties of the generalized Cartan matrices, interpreted as a matrix of divisor intersections.

3. From Berger graphs to Berger matrices, a algebra review

Once one has established the existence of Dynkin-like graphs, possibly not corresponding to any of the known Lie or affine Kac-Moody algebras, the next step is to encode the information contained in the graphs in a more workable structure: a matrix of integer numbers to be defined. If these “Dynkin” graphs are somehow related to possible generalizations of the Lie and affine Kac-Moody algebra concepts, it is then natural to look for possible generalizations of the corresponding affine Kac-Moody Cartan matrices when searching for possible ways of assigning integral matrices to them. We include here a little review of some definitions and the procedure of the formal definition of Berger matrices already outlined in Ref.[16]. There we mention one possibility which could serve of guide: to suppose that this affine property remains: matrices with determinant equal to zero and all principal minors positive. We will see in what follows that this is a sensitive choice, on the other hand it turn out that the usual conditions on the value of the diagonal elements has to be abandoned.

In first place, the building of Cartan-like matrices from already existing graphs is as follows. We assign to any generalized Dynkin diagram, a set of vertices and lines connecting them, a matrix, B , whose non diagonal elements are either zero or are negative integers. There are different possibilities, for non diagonal elements, considering for the moment the most simple case of “laced” graphs leading to symmetric assignments, we have: Case A) there is no line from the vertex i to the vertex j . In this case the element of the matrix $B_{ij} = 0$. Case B) there is a single line connecting $i - j$ vertices. In this case $B_{ij} = -1$.

The diagonal entries should be defined in addition. As a first step, no special restriction is applied and any positive integer is allowed. We see however that very quickly only a few possibilities are naturally selected. The diagonal elements of the matrix are two for CY2 originated graphs but are allowed to take increasing integer numbers with the dimensionality of the space, 3, 4... for $CY_{3,4,...}$.

A large number of graphs and matrices associated to them, obtained by inspection considering different possibilities has been checked (see also Ref.[9]). Some regularities are quickly disclosed. In first place it is easy to see that there are graphs where the number of lines outgoing a determined vertex can be bigger than two, in cases of interest they will be 3, graphs from CY3, or bigger in the cases of graphs coming from CY4 and higher dimensional spaces. Some other important regularities appear. The matrices are genuine generalizations of affine matrices. Their determinant can be made equal to zero and all

their principal minors made positive by careful choice of the diagonal entries depending on the Calabi-Yau dimension and n-ary structure.

Moreover, we can go back to the defining reflexive polyhedra and define other quantities in purely geometrical terms. For example we can consider the position or distance of each of the vertices of the generalized Dynkin diagram to the intersecting reflexive polyhedra. Indeed, it has been remarked [5] that Coxeter labels for affine Kac-Moody algebras can be obtained directly from the graphs: they correspond precisely to this “distance” between individual nodes and some defined intersection which separates “left” and “right” graphs. Intriguingly, this procedure can be easily generalized to our case, one can see that, by a careful choice of the entry assignment for the corresponding matrix, it follows Coxeter labels can be given in a proper way: they have the expected property of corresponding to the elements of the null vector a generalized Cartan matrix.

From the emerging pattern of these regularities, we are lead to define a new set of matrices, generalization of Cartan matrices in purely algebraic terms, the Berger, or Berger-Cartan-Coxeter matrices. This will be done in the next paragraph.

Based on previous considerations, we define now in purely algebraic terms [16], the so called Berger Matrices [9, 16]. We suggest the following rules for them, in what follows we will see step by step how they lead to a consistent construction generalizing the Affine Kac-Moody concept. A Berger matrix is a finite integral matrix characterized by the following data:

$$\begin{aligned}\mathbb{B}_{ii} &= 2, 3, 4.. \\ \mathbb{B}_{ij} &\leq 0, \quad \mathbb{B}_{ij} \in \mathbb{Z}, \\ \mathbb{B}_{ij} = 0 &\mapsto \mathbb{B}_{ji} = 0, \\ Det \mathbb{B} &= 0, \\ Det \mathbb{B}_{\{(i)\}} &> 0.\end{aligned}$$

The last two restrictions, the zero determinant and the positivity of all principal proper minors, corresponds to the *affine condition*. They are shared by Kac-Moody Cartan matrices, so we expect that the basic definitions and properties of those can be easily generalized. However, with respect to them, we relaxed the restriction on the diagonal elements. Note that, more than one type of diagonal entry is allowed: 2, 3, .. diagonal entries can coexist in a given matrix.

For the sake of convenience, we define also “*non-affine*” *Berger Matrices* where the condition of non-zero determinant is again imposed. These matrices does not seem to appear naturally resulting from polyhedron graphs but they are useful when defining root systems for the affine case by extension of them. They could play the same role of basic simple blocks as finite Lie algebras play for the case of affine Kac-Moody algebras.

The important fact to be remarked here is that this definition lead us to a construction with the right properties we would expect from a generalization of the Cartan matrix idea.

The systematic enumeration of the various possibilities concerning the large family of possible Berger matrices can be facilitated by the introduction for each matrix of its generalized Dynkin diagram. As we intend that the definition of this family of matrices

be independent of algebraic geometry concepts we need an independent definition of these diagrams. Obviously the procedure given before can be reversed to allow the deduction of the generalized Dynkin diagram from its generalized Cartan or Berger Matrix. An schematic prescription for the most simple cases could be: A) For a matrix of dimension n , define n vertices and draw them as small circles. In case of appearance of vertices with different diagonal entries, some graphical distinction will be performed. Consider all the element i, j of the matrix in turn. B) Draw one line from vertex i to vertex j if the corresponding element A_{ij} is non zero.

In what follows, we show that indeed these kind of matrices and Dynkin diagrams, exist beyond those purely defined from Calabi-Yau newton reflexive polyhedra. In fact we show that there are infinity families of them where suggestive regularities appear.

It seems easy to conjecture that the set of all, known or generalized, Dynkin diagrams obtained from Calabi-Yau spaces can be described by this set of Berger matrices. It is however not so clear the validity of the opposite question, whether or not the infinite set of generalized Dynkin diagrams previously defined can be found digging in the Calabi-Yau (n, a) structure indicated by UCYA. For physical applications however it could be important the following remark. Theory of Kac-Moody algebras show us that for any *finite* or *affine* Kac-Moody algebra, every proper subdiagram (defined as that part of the generalized Coxeter-Dynkin diagram obtained by removing one or more vertices and the lines attached to these vertices) is a collection of diagrams corresponding to *finite* Kac-Moody algebras. In our case we have more flexibility. Proper subdiagrams, obtained eliminating internal nodes or vertices, are in general collections of Berger-Coxeter-Dynkin diagrams corresponding to other (affine by construction)Berger diagrams *or* to *affine* Kac-Moody algebras. This property might open the way to the consideration of non-trivial extensions of SM and string symmetries.

Next, one consider the Berger Matrix as a matrix of inner products in some root spaces. Moreover, for further progress, the interpretation of a Berger matrix as the matrix of divisor intersections $B_{ij} \sim D_i \cdot D_j$ in Toric geometry could be useful for the study of the viability of fans of points associated to them, singularity blow-up, and the existence of Calabi-Yau varieties itself. This geometrical approach will be pursued somewhere else [26]. However, for algebraic applications, and with the extension of the CLA and KMA concepts in mind, the interpretation of these matrices as matrices corresponding to a inner product in some vector space is most natural which is our objective now.

The Berger matrices are obtained by weakening the conditions on the generalized Cartan matrix \hat{A} appearing in affine Kac-Moody algebras. In what concern algebraic properties, there are no changes, it remains intact the condition of semi-definite positiveness, this allows to translate trivially many of the basic ideas and terminology for roots and root subspaces for appearing in Kac-Moody algebras. Clearly, the problem of expressing the “simple” roots in a orthonormal basis was an important step in the classification of semisimple Cartan-Lie algebras.

For a Berger matrix B_{ij} of dimension n , the rank is $r = n - 1$. The $(r + 1) \times (r + 1)$ dimensional is nothing else that a generalized Cartan matrix. This matrix is symmetric in all the cases of interest in this work. We expect that a simple root system $\Delta^0 = \{\alpha_1, \dots, \alpha_r\}$

and an extended root system by $\hat{\Delta}^0 = \alpha_0, \alpha_1, \dots, \alpha_r$, can be constructed. The defining relation is that the (scaled) inner product of the roots is

$$\alpha_i \cdot \alpha_j = \hat{\mathbb{B}}_{ij} \quad 1 \leq i, j \leq n. \quad (3.1)$$

The set of roots α_i are the simple roots upon which our generalized Cartan Matrix is based. They are supposed to play the analogue of a root basis of a semisimple Lie Algebra or of a Kac-Moody algebra. Note that, as happens in KMA Cartan matrices, for having the linearly independent set of α_i vectors, we generically define them in, at least, a $2n - r$ dimensional space H . In our case, as $r = n - 1$, we would need a $n + 1$ dimensional space. Therefore, the set of n roots satisfying the conditions above has to be completed by some additional vector, the “null root”, to obtain a basis for H . The consideration of these complete set of roots will appear in detail elsewhere [26].

A generic root, α , has the form

$$\alpha = \sum_i c_i \alpha_i$$

where the set of the coefficients c_i are either all non-negative integers or all non-positive integers. In this $n + 1$ dimensional space H , generic roots can be defined and the same generalized definition for the inner product of two generic roots α, β as in affine Kac-Moody algebras applies. This generalized definition reduces to the inner product above for any two simple roots.

Since B is of rank $r = n - 1$, we can find one, and only one, non zero vector μ such that

$$B\mu = 0.$$

The numbers, a_i , components of the vector μ , are called Coxeter labels. The sums of the Coxeter labels $h = \sum \mu_i$ is the Coxeter number. For a symmetric generalized Cartan matrix only this type of Coxeter number appear.

to modify? XXX

For each affine matrix we can obtain a number of non-equivalent derived non affine matrix of dimension of smaller dimension simply by eliminating one or more of the columns and rows. In terms of the graph, this correspond to the elimination of any one of the nodes. We can explicitly check in all the cases that the determinant of these matrices are strictly positive and that the matrices are positive definite. We can in the same way write the set of roots $\alpha_i, i = 1, \dots, 12$ for this non affine matrix B^{n-aff} such that $B_{i,j}^{n-aff} = \alpha_i \cdot \alpha_j$. New vectors, fundamental weights, that will play an important role later are These fundamental weights are defined as the vectors $\Lambda_i, i = 1, \dots, 12$ such that $\delta_{ij} = \Lambda_i \cdot \alpha_j$. In the basis of the α'_i s they are basically given by the coefficients of the inverse of the non-affine matrix B^{n-aff} .

4. The simply laced cases

Let consider the reflexive polyhedron, which corresponds to a K3-sliced CY_3 space and which is defined by two extended vectors [9] $\vec{k}_L^{ext}, \vec{k}_R^{ext}$. One of these vectors is coming from

the set $S_L = \{(0, 0, 0, 0, \vec{k}_1), (0, 0, 0, 0, \vec{k}_2), (0, 0, 0, 0, \vec{k}_3), \dots (perms)..\}$, where the remaining dots correspond to permutations of the position of zeroes and vectors k , for example permutations of the type $\{(0, k, 0, 0, 0), (k, 0, 0, 0, 0), \text{etc}\}$. The other defining vector can come from the set $S_R = \{(0, \vec{k}_4), \dots (perms)..\}$. The vectors $\vec{k}_1, \vec{k}_2, \vec{k}_3$ are respectively any of the five RWVs of dimension 1, 2 and 3. The vector \vec{k}_4 correspond to any of the 95 K3 RWVs of dimension four. As a simple example, a generic quintic CY3 can be defined by two extended vectors, $\vec{k}_{1L}^{(ext)} = (1, 0, 0, 0, 0)$ and $\vec{k}_{2R}^{(ext)} = (0, 1, 1, 1, 1)$ (which correspond to the choice $\vec{k}_4 = (1, 1, 1, 1)$). The left and right skeletons of the reflexive polyhedron are determined by extended vectors, $\vec{k}_{1L}^{(ext)}, \vec{k}_{2R}^{(ext)}$ respectively. The left skeleton will be a tetrahedron with 4-vertices, 6 edges and a number of internal points over the edges as indicated in the Figure 3 of Ref.[9].

The RW-simply-laced vectors for dimension 1, 2 and 3 and their graphs have already been considered before, there are five and only five cases:

- dim =1 the vector (1)[1] which can be associated to the A series of Dynkin diagrams,
- dim=2, we have the vector (11)[2], which is associated to the D series,
- and dim=3, where the set of vectors (111)[3], (112)[4], (123)[6], correspond, as firstly shown by Candelas and Font, to the affine exceptional algebras $E_{6,7,8}^{(1)}$.

The main objective of this work is to enlarge this list with graphs obtained by vectors of dimension four (corresponding to CY3). In dim=4, corresponding to K3-sliced CY₃ spaces, we can single out by inspection the following 14 RW-reflexive vectors from the total of 95- K3-vectors (1111), (1122), (1113), (1124), (2334), (1344), (1236), (1225), (14510), (1146), (1269), (1,3,8,12), (2,3,10,15), (1,6,14,21). The graphs corresponding to these vectors can easily be obtained as explained before. From the geometrical construction Coxeter numbers can be easily assigned to each of the nodes of these graphs. Moreover we can assign genuine Berger matrices to them with specially simple properties: they are symmetric, affine (the determinant is zero, the rank one less than the dimension), they lead to the same set of Coxeter as those obtained from the geometrical construction. In addition, each of these graphs and matrices seems not be “extendable”: in contradistinction to other cases, see the discussion in Ref.[16], no other graphs and Berger matrices can be obtained from them simply adding more nodes to any of the legs. In this sense, these graphs are “exceptional”. As with the classical exceptional graphs, series can be traced among them. Apparently these fourteen vectors are the only ones from the total of 95 vectors which lead to this kind of symmetric matrices.

4.1 The exceptional simply laced graphs from CY3

The graphs and matrices of these simply laced graphs, both, those already known of dimension 1, 2, 3 and those new of dimension 4 share a number of simple characteristics. The cases of dimension 1, 2, 3 are well known and correspond to the classical Cartan Lie algebras. Our objective is to give a general description of the new graphs. The Berger matrix is obtained from the planar graph according to the standard rules. We assign different values (2 or 3

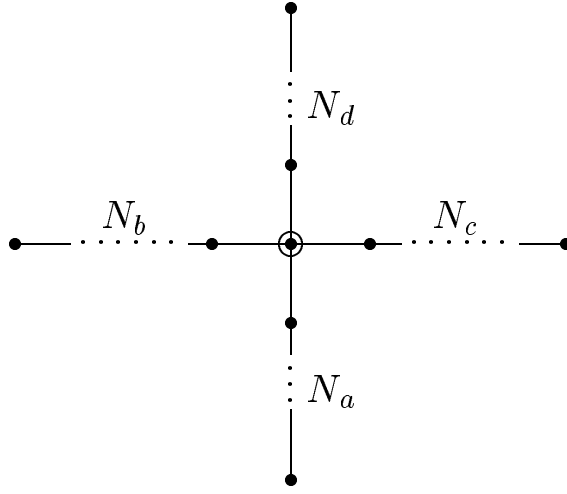


Figure 2: The general graph for simply laced cases.

) to diagonal entries depending if they are associated to standard nodes or to the central vertex. One can assign to all of these new graphs a Berger matrix with the following block structure:

$$B_{SL} = \begin{pmatrix} A & 0 & 0 & 0 & v_1 \\ 0 & B & 0 & 0 & v_2 \\ 0 & 0 & C & 0 & v_3 \\ 0 & 0 & 0 & D & v_4 \\ v_1^t & v_2^t & v_3^t & v_4^t & 3 \end{pmatrix} \quad (4.1)$$

where A, B, C, D are square matrices of various dimensions with diagonals filled with two, they are the equivalent of the A_r Cartan matrices and the v_i column vectors filled with zeroes except for one negative entry, $v_i^t = (0, \dots, 0, -1)$.

A generic graph for anyone of these fourteen vectors is of the form depicted in Fig.(2). As we can see in this figure from a central node four legs with respectively (N_a, N_b, N_c, N_d) nodes are attached. Each of the legs correspond to one of the regular blocks of the Berger matrix SL . The central node correspond to the one dimensional block filled with 3 in the matrix. The Coxeter labels can given in a systematic way [9, 16], they agree with those directly obtained from the matrix B_{SL} . Non affine matrices can be obtained eliminating one or more nodes from the legs. Clearly the number of non-affine matrices depends on the number of eliminated nodes and on the symmetry of the diagram. In what follows we will list all the non-affine matrices of dimension one less of the original matrix. In all these cases can be explicitly checked that the matrices are strictly positive definite.

For each diagram, system of roots α_i , set of vectors which realize the Berger matrix SL as a matrix of scalar products, can be easily obtained. Given a minimal set of orthonormal canonical vectors $\{e_{ai}\}$, one can consider roots of the form, for each of the legs

$$\begin{aligned} \alpha_{ai} &= e_{ai} - e_{a,i+1} \\ \alpha_{bi} &= e_{bi} - e_{b,i+1} \end{aligned}$$

$$\alpha_{ci} = e_{ci} - e_{c,i+1}$$

$$\alpha_{di} = e_{di} - e_{d,i+1}$$

where the set of roots and vectors are assigned for each of the legs $l = a, b, c, d$ for the sake of clarity. The root corresponding central node, the one corresponding to the entry 3 in the matrix, is assigned $\alpha_{central} = -(\alpha_{a1} + \alpha_{b1} + \alpha_{c1} + \alpha_{d1})$, and α_{l1} are roots corresponding to the nearest nodes. The affine condition is the used to reduce the dimensionality of the space spanned by the e'_i 's. The dimensionality of this space can be furtherly reduced. This can be systematically done in a number of ways: we can write one,two or more roots as a lineal combination of the rest of them with unknown coefficients and ask for the scalar products relations to be fulfilled. For the sake of simplicity let us take as a representative example any of the cases where one of the legs has only one node, i.e. $N_d = 1$. We can write

$$\begin{aligned}\alpha_{ai} &= e_{ai} - e_{a,i+1}, \quad i = 1, \dots, N_a - 1 \\ \alpha_{bi} &= e_{bi} - e_{b,i+1}, \quad i = 1, \dots, N_b \\ \alpha_{ci} &= e_{ci} - e_{c,i+1}, \quad i = 1, \dots, N_c - 1 \\ \alpha_{cent} &= -(e_{a1} + e_{b1} + e_{c1}).\end{aligned}\tag{4.2}$$

The two roots $\alpha_{a,N_a}, \alpha_{c,N_c}$, are obtained by imposing the scalar products conditions.

$$\begin{aligned}\alpha_{a,N_a} \cdot \alpha_{a,N_a-1} &= -1, \quad \alpha_{a,N_a}^2 = 2, \\ \alpha_{c,N_c} \cdot \alpha_{c,N_c-1} &= -1, \quad \alpha_{c,N_c}^2 = 2\end{aligned}$$

and the mixed relation

$$\alpha_{a,N_a} \cdot \alpha_{c,N_c} = 0.$$

The root corresponding to the fourth leg, α_{d1} , is obtained by imposing the affine condition at the end of the procedure. The coefficients of the affine condition are the Coxeter labels and these are known from the beginning by condition. One can check that the following expression with arbitrary coefficients x_j satisfy automatically the first condition $\alpha_{a,N_a} \cdot \alpha_{a,N_a-1} = -1$,

$$\alpha_{a,N_a} = \frac{1}{x_1 + x_2} \left(x_1 e_{c,N_c} - x_2 \sum_1^{N_c-1} e_{ci} + x_3 \sum_1^{N_b+1} e_{bi} + x_4 \sum_1^{N_a} e_{ai} \right)$$

and similarly for α_{c,N_c} with arbitrary coefficients y_j :

$$\alpha_{c,N_c} = \frac{1}{y_1 + y_2} \left(y_1 e_{a,N_a} - y_2 \sum_1^{N_a-1} e_{ai} + y_3 \sum_1^{N_b+1} e_{bi} + y_4 \sum_1^{N_c} e_{ci} \right).$$

These 4 + 4 coefficients are constrained from three non-linear equations obtained from the other scalar products.

$$\begin{aligned}2(x_1 + x_2) &= x_1^2 + (N_c - 1)x_2^2 + (N_b + 1)x_3^2 + N_a x_4^2 \\ 2(y_1 + y_2) &= y_1^2 + (N_a - 1)y_2^2 + (N_b + 1)y_3^2 + N_c y_4^2 \\ 0 &= x_1 y_4 + x_4 y_1 - (N_c - 1)x_2 y_4 + (N_b + 1)x_3 y_3 - (N_a - 1)x_4 y_2.\end{aligned}$$

Solutions to equations of this type are obtained for a number of cases presented on continuation.

We present in table (1) the list of all the 14 RW vectors of dimension four give above. This table contains all the necessary information to write down the matrices and graphs for each case. For each of the vectors we give the list of integers (N_a, N_b, N_c, N_d) which define both, the number of nodes in each of the four legs of the graph, see Fig.(2), and the dimension of the each of the block matrices A, B, C, D . In Figs.(5-8) we explicitly give all the graphs with their Coxeter Labels. On continuation we include in the table, the dimension of the graph or the matrix, which is equal to the rank plus one and the Coxeter number h which is the sum the list of the Coxeter labels given in the next entry. The last two entries of the table contain information about the non-affine derived matrices. The first number is the number of non equivalent non-matrices of maximal dimension which can be obtained eliminating one of the nodes of the graph (or just one column and row in the respective matrix), the second number is the smallest determinant of any of these non-affine derived matrices. We note that the dimensions of the affine matrices are well bounded in the range $dim \sim (10, 50)$ just above the characteristic dimension of the standard affine algebras $E^{(1)}$. We also note that the total number of non-affine matrices obtained from these 14 simply laced cases is 34. The list of the dimensions of these matrices are $(12, 14, 16, 18, 20, 25, 26, 27, 28, 33, 50)$ where two series of five and four members can be recognized in addition to two isolated dimensions. It could be instructive to compare the values of the determinants of these non-affine cases with the values for the determinants of the Cartan matrices of the well-known non-affine Cartan-Lie algebras $\det(E_{6,7,8}) = 3, 2, 1, \det(F_4, G_2) = 1, \det(B_r, C_r) = 2, \det D_r = 4, \det A_r = r + 1$.

We could ask the question on how we could enlarge this list of affine graphs and matrices using our Berger construction. Following Ref.[16], new matrices could be obtained for example starting from any of these graph and inserting additional internal nodes. However these affine matrices seems to be exceptional, no other affine matrices can be obtained from them in this way.

One can also ask the question wether among the graphs and matrices presented in table (1) one can find some series in a similar way as the $E_{6,7,8}$ seem to form a series. Candidates for series like that are the graphs of consecutive dimension $(13, 14, 15)$ and those of the list $(26, 27, 28, 29)$. Indeed one can see that the cases of dimension 13, 14, 15 present some similitudes to the $E_{6,7,8}$ series, in particular the root systems of the vectors (1122) and (2334) are related to each other in a similar way as the E_7^1, E_8^1 roots are linked.

In the next paragraphs we will deal in some more detail with each of the fourteen cases in turn, paying some more attention to the cases corresponding to the cases of of lower dimension 13, 14, 15.

4.2 Example: the (1111) case.

We discuss in some detail the first case apearing in table(2), the matrix associated to the vector (1111) [4]. The Berger matrix is obtained from the planar graph according to the standard rules. We assign different values (2 or 3) to diagonal entries depending if they are associated to standard nodes or to the central vertex. The result is the following 13×13

Vector \vec{k}_4	N_a, N_b, N_c, N_d	Dim	h	$(\dots, h_i, dots)$	N_{Aff}	Det
(1, 1, 1, 1)[4]	(3,3,3,3)	13	28	(3, 2, 1, 3, 2, 1, 3, 2, 1, 3, 2, 1, 4).	1	16
(2, 3, 3, 4)[12]	(2,3,3,5)	14	90	(4, 8, 3, 6, 9, 3, 6, 9, 2, 4, ..., 12)	3	8
(1, 1, 1, 3)[6]	(1,5,5,5)	17	54	(3, 1, 2, ..., 5, 1, 2, ..., 5, 1, 2, ..., 6)	1	12
(1, 1, 2, 2)[6]	(2,2,5,5)	15	48	(2, 4, 2, 4, 1, 2, ..., 5, 1, 2, ..., 6)	2	9
(1, 1, 2, 4)[8]	(1,3,7,7)	19	80	(4, 2, 4, 6, 1, 2, ..., 7, 1, 2, ..., 8)	2	8
(1, 2, 2, 5)[10]	(1,4,4,9)	19	100	(5, 2, 4, 6, 8, 2, 4, 6, 8, 1, 2, ..., 10)	2	5
(1, 2, 3, 6)[12]	(1,3,5,11)	21	132	(6, 3, 6, 9, 2, 4, ..., 10, 1, 2, ..., 12)	3	6
(1, 3, 4, 4)[12]	(2,2,3,11)	19	120	(4, 8, 4, 8, 3, 6, 9, 1, 2, ..., 12)	3	3
(1, 4, 5, 10)[20]	(1,3,4,19)	28	290	(10, 5, 10, 15, 4, 8, ..., 16, 1, 2, ..., 20)	3	2
(1, 1, 4, 6)[12]	(1,2,11,11)	26	162	(6, 4, 8, 1, 2, ..., 11, 1, 2, ..., 12)	2	6
(1, 2, 6, 9)[18]	(1,2,8,17)	29	270	(9, 6, 12, 2, 4, ..., 16, 1, 2, ..., 18)	3	3
(1, 3, 8, 12)[24]	(1,2,7,23)	34	420	(12, 8, 16, 3, 6, ..., 21, 1, 2, ..., 24)	3	2
(2, 3, 10, 15)[30]	(1,2,9,14)	27	420	(5, 10, 20, 3, 6, ..., 27, 2, 4, ..., 30)	3	4
(1, 6, 14, 21)[42]	(1,2,6,41)	51	1092	21, 14, 28, 6, 12, ..., 36, 1, 2, ..., 42)	3	1

Table 1: List of all the 14 RW vectors of dimension four. The integers (N_a, N_b, N_c, N_d) define both, the number of nodes in each of the four legs of the graph and the dimension of the each of the block matrices A, B, C, D . The dimension of the matrix ($dim = rank + 1$). The Coxeter number h and list of Coxeter labels.. The last two entries of correspond to the non-affine derived matrices. First, the number of non equivalent non-matrices of maximal dimension which can be obtained eliminating one of the nodes of the graph (or just one column and row in the respective matrix), the second number is the smallest determinant of any of these non-affine derived matrices.

symmetric matrix containing, as more significant difference, an additional 3 diagonal entry:

$$CY3B_0 = \left(\begin{array}{ccc|ccc|ccc|c} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ \hline -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 3 \end{array} \right) \quad (4.3)$$

One can check that this matrix fulfills the conditions for Berger matrices. Its determinant is zero while the rank $r = 12$. All the principal minors are positive.

One can obtain a system of roots $(\alpha_i, i = 1, \dots, 13)$ in a orthonormal basis. Considering the orthonormal canonical basis $(\{e_i\}, i = 1, \dots, 12)$, we obtain:

$$\begin{aligned} \alpha_1 &= -(e_1 - e_2) \\ \alpha_2 &= \frac{1}{2}[(e_1 - e_2 - e_3 + e_4 + e_5 + e_6) + (e_8 - e_7)] \end{aligned}$$

$$\begin{aligned}
\alpha_3 &= -(e_8 - e_7) \\
\alpha_4 &= (e_4 - e_3) \\
\alpha_5 &= (e_5 - e_4) \\
\alpha_6 &= (e_6 - e_5) \\
\alpha_7 &= (e_1 + e_2) \\
\alpha_8 &= -\frac{1}{2}[(e_1 + e_2 - e_9 - e_{10} - e_{11} - e_{12}) + (e_8 + e_7)] \\
\alpha_9 &= (e_8 + e_7) \\
\alpha_{10} &= -(e_{10} - e_9) \\
\alpha_{11} &= -(e_{11} - e_{10}) \\
\alpha_{12} &= -(e_{12} - e_{11}) \\
\alpha_{13} &= e_3 - e_2 - e_9
\end{aligned}$$

The assignment of roots to the nodes of the Berger-Dynkin graph is given in Fig.(4). It easily to check the inner product of these simple roots leads to the Berger Matrix $a_i \cdot a_j = B_{ij}$. This matrix has one null eigenvector, with coordinates, in the α basis, $\mu = (3, 2, 1, 3, 2, 1, 3, 2, 1, 3, 2, 1, 4)$. The Coxeter number is $h = 22$. One can check that these Coxeter labels are identical to those obtained from the geometrical construction [5, 9]. They are shown explicitly in Fig.(3). Correspondingly the following linear combination of the roots satisfies the affine condition:

$$4\alpha_0 + 3\alpha_{a1} + 2\alpha_{a2} + \alpha_{a3} + 3\alpha_{b1} + 2\alpha_{b2} + \alpha_{b3} + 3\alpha_{c1} + 2\alpha_{c2} + \alpha_{c3} = 0$$

It is instructive to compare this case with the standard $E_6^{(1)}$ case. The graph associated to this case can be extracted from a (111) reflexive Newton polyhedron. The result appears in Fig.(3,center), we obtain the Coxeter-Dynkin diagram corresponding to the affine algebra $E_6^{(1)}$. We can easily check that following the rules given above we can form an associated Berger matrix, which, coincides with the corresponding generalized Cartan matrix of the the affine algebra $E_6^{(1)}$. The well known Cartan matrix for this is:

$$E_6^{(1)} \equiv CYB3 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 0 & -1 & 2 \end{pmatrix} \quad (4.4)$$

The root system is well known, we have (in a, minimal, ortonormal basis ($\{e_i\}, i = 1, \dots, 8$):

$$\begin{aligned}
\alpha_1 &= -\frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 - e_8) \\
\alpha_2 &= (e_2 - e_1) \\
\alpha_3 &= (e_4 - e_3) \\
\alpha_4 &= (e_5 - e_4)
\end{aligned}$$

$$\begin{aligned}
\alpha_5 &= (e_1 + e_2) \\
\alpha_6 &= -\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8) \\
\alpha_7 &= -e_2 + e_3
\end{aligned}$$

Coxeter labels and affine condition are easily reobtained. The diagonalization of the matrix gives us the zero mode vector, $B\mu = 0$. In this case the Coxeter labels are $\mu = (1, 2, 1, 2, 1, 2, 3)$ and $h = 12$. The affine condition satisfied by the set of simple roots is also well known $\alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 + 3\alpha_7 = 0$.

From the (1111)[4] *CY3B* affine matrix we can obtain a derived non affine matrix of dimension 12 simply by eliminating one of the columns and rows. It is straightforward to write the graph for it. It is obvious for symmetry reasons that in this case there is only one such affine matrix. We can explicitly check that the determinant of this matrix is strictly positive ($\det(BE_6) = 16$). Furthermore we have checked that the matrix is positive definite. We can in the same way write the set of twelve roots $\alpha_i, i = 1, \dots, 12$ for this non affine matrix B^{n-aff} such that $B_{i,j}^{n-aff} = \alpha_i \cdot \alpha_j$. The fundamental weights $\Lambda_i, i = 1, \dots, 12$ satisfy $\delta_{ij} = \Lambda_i \cdot \alpha_j$. In the basis of the α_i 's they can be obtained from the inverse of the non-affine matrix B^{n-aff} . The coefficients of fundamental weights Λ_i in this basis are given in the next table:

<i>F.W.</i>	α_{a1}	α_{a2}	α_{b1}	α_{b2}	α_{b3}	α_{c1}	α_{c2}	α_{c3}	α_{d1}	α_{d2}	α_{d3}	α_0
Λ_{a1}	6	3	6	4	2	6	4	2	6	4	2	8
Λ_{a2}	3	2	3	2	1	3	2	1	3	2	1	4
Λ_{b1}	6	3	15/2	5	5/2	27/4	9/2	9/4	27/4	9/2	9/4	9
Λ_{b2}	4	2	5	4	2	9/2	3	3/2	9/2	3	3/2	6
Λ_{b3}	2	1	5/2	2	3/2	9/4	3/2	3/4	9/4	3/2	3/4	3
Λ_{c1}	6	3	27/4	9/2	9/4	15/2	5	5/2	27/4	9/2	9/4	9
Λ_{c2}	4	2	9/2	3	3/2	5	4	2	9/2	3	3/2	6
Λ_{c3}	2	1	9/4	3/2	3/4	5/2	2	3/2	9/4	3/2	3/4	3
Λ_{d1}	6	3	27/4	9/2	9/4	27/4	9/2	9/4	15/2	5	5/2	9
Λ_{d2}	4	2	9/2	3	3/2	9/2	3	3/2	5	4	2	6
Λ_{d3}	2	1	9/4	3/2	3/4	9/4	3/2	3/4	5/2	2	3/2	3
Λ_0	8	4	9	6	3	9	6	3	9	6	3	12

(4.5)

One could try to pursue the generalization process of graphs and matrices adding internal nodes to this case as it has been done previously. Surprisingly, in contradiction to previous case where an infinite series of new graphs and matrices can be obtained [16], this is however an “exceptional” case. No infinite series of graphs can be obtained in this way. Similarly, one can find generalizations of the $E_7^{[1]}$ and $E_8^{[1]}$ graphs (corresponding to the choice of three dimensional vectors (112),(123)).

4.3 Example: the (1122)(6) case

In the next example, one constructs the Berger matrix and graph based of the vector $\vec{k}_4 = (1122)[6]$ from *CY*₃. The graph associated to this vector appears in Fig.(5,left). The Berger matrix is obtained from the planar graph according to the standard rules. We assign different values (2 or 3) to diagonal entries depending if they are associated to standard

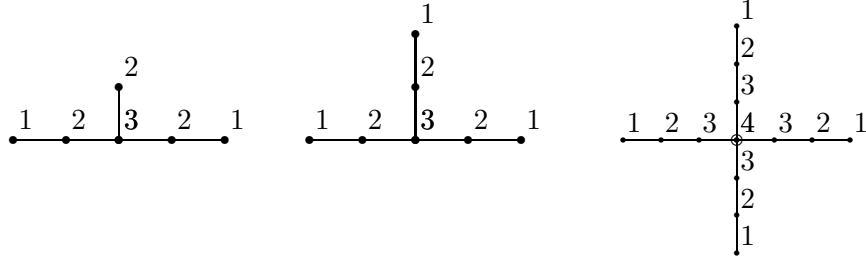


Figure 3: Berger-Dynkin diagrams for E_6 , the affine $E_6^{(1)}$ and its generalization $CY3 - E_6^{(1)}$.

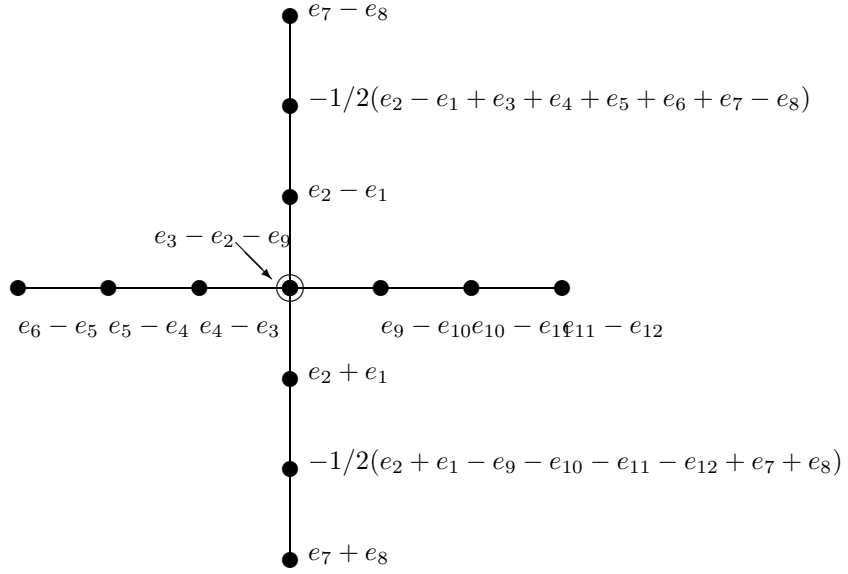


Figure 4: Berger-Dynkin diagram and root system for the $CY3 - E_6^{(1)}$ matrix.

nodes or to the central vertex. The result is the following 15×15 symmetric matrix

$$CY3B(1122) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \end{pmatrix} \quad (4.6)$$

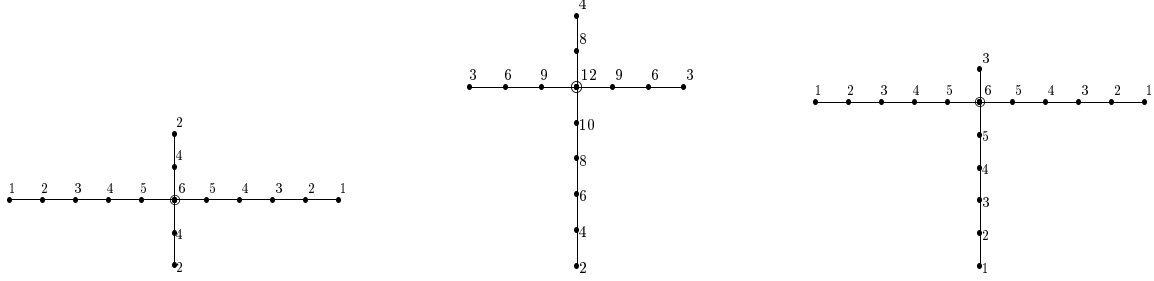


Figure 5: Berger Graphs corresponding to the vectors (from left to right) $(1122)[6]$, $(2334)[12]$ and $(1113)[6]$. The Coxeter labels for each node are presented.

One can check that this matrix fulfills the conditions for Berger matrices. Its determinant is zero while the rank $r = 14$. One can obtain a system of roots $(\alpha_i, i = 1, \dots, 15)$ in a orthonormal basis. Considering the orthonormal canonical basis $(\{e_i\}, i = 1, \dots, 14)$, we obtain:

$$\begin{aligned}
\alpha_{b5} &= e_{13} - e_{14} \\
&\dots \\
\alpha_{b1} &= e_9 - e_{10} \\
\alpha_{c5} &= e_8 - e_7 \\
&\dots \\
\alpha_{c1} &= e_4 - e_3 \\
\alpha_{a2} &= 1/2(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8) \\
\alpha_{a1} &= e_2 - e_1 \\
\alpha_{d2} &= -1/2(e_1 + e_2 - e_9 - e_{10} - e_{11} - e_{12} - e_{13} - e_{14}) \\
\alpha_{d1} &= e_2 + e_1 \\
\alpha_0 &= e_3 - e_2 - e_9
\end{aligned}$$

The assignment of roots to the nodes of the Berger-Dynkin graph is given according to the notation of Fig.(2). It is easy to check the inner product of these simple roots leads to the Berger Matrix $a_i \cdot a_j = B_{ij}$. This matrix has one null eigenvector, with coordinates, in the α basis, $\mu = (2, 4, 2, 4, 1, 2, \dots, 5, 1, 2, \dots, 6)$. The Coxeter number is $h = 48$. One can check that these Coxeter labels are identical to those obtained from the geometrical construction [5, 9]. They are shown explicitly in Fig.(5, left). Correspondingly the following linear combination of the roots satisfies the affine condition:

$$6\alpha_0 + 2\alpha_{a1} + 4\alpha_{a2} + 2\alpha_{b1} + 4\alpha_{b2} + 1\alpha_{c1} + 2\alpha_{c2} + \dots + 5\alpha_{c3} + 1\alpha_{d1} + 2\alpha_{d2} + \dots + 5\alpha_{d3} = 0$$

For this affine matrix we can obtain *two* non-equivalent derived non affine matrix of dimension 14 simply by eliminating one of the columns and rows. In terms of the graph, this corresponds to the elimination of any one of the nodes labeled with Coxeter labels 1 or 2. We can explicitly check that the determinant of these matrices are strictly positive

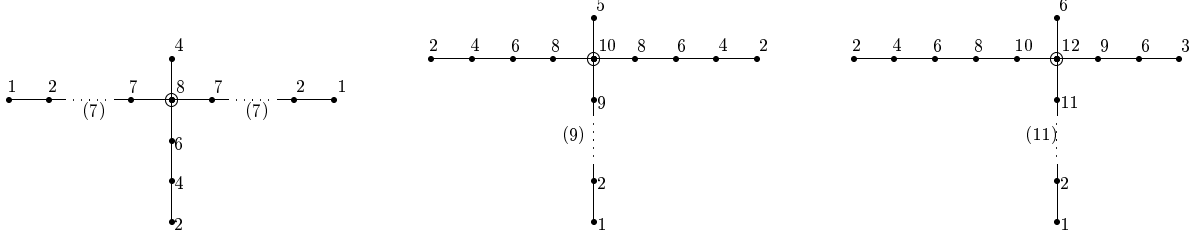


Figure 6: Berger Graphs corresponding to the vectors (from left to right) $(1124)[8]$, $(1225)[10]$ and $(1236)[12]$.

($\det = 10, 32$ for elimination Coxeter label nodes 1, 2 respectively). Furthermore we have checked that the matrices are positive definite. We can in the same way write the set of roots $\alpha_i, i = 1, \dots, 12$ for this non affine matrix B^{n-aff} such that $B_{i,j}^{n-aff} = \alpha_i \cdot \alpha_j$. The fundamental weights are defined as before. The ones corresponding to the elimination of the Coxeter label 1 node are given in table(2).

	α_{b4}	α_{b3}	α_{b2}	α_{b1}	α_{c5}	α_{c4}	α_{c3}	α_{c2}	α_{c1}	α_{a3}	α_{a2}	α_{d3}	α_{d2}	α_0
Λ_{b4}	2	3	4	5	1	2	3	4	5	2	4	2	4	6
Λ_{b3}	3	6	8	10	2	4	6	8	10	4	8	4	8	12
Λ_{b2}	4	8	12	15	3	6	9	12	15	6	12	6	12	18
Λ_{b1}	5	10	15	20	4	8	12	16	20	8	16	8	16	24
Λ_{c5}	1	2	3	4	5/3	7/3	3	11/3	13/3	5/3	10/3	5/3	10/3	5
Λ_{c4}	2	4	6	8	7/3	14/3	6	22/3	26/3	10/3	20/3	10/3	20/3	10
Λ_{c3}	3	6	9	12	3	6	9	11	13	5	10	5	10	15
Λ_{c2}	4	8	12	16	11/3	22/3	11	44/3	52/3	20/3	40/3	20/3	40/3	20
Λ_{c1}	5	10	15	20	13/3	26/3	13	52/3	65/3	25/3	50/3	25/3	50/3	25
Λ_{a3}	2	4	6	8	5/3	10/3	5	20/3	25/3	4	7	10/3	20/3	10
Λ_{a2}	4	8	12	16	10/3	20/3	10	40/3	50/3	7	14	20/3	40/3	20
Λ_{d3}	2	4	6	8	5/3	10/3	5	20/3	25/3	4	7	10/3	20/3	10
Λ_{d2}	4	8	12	16	10/3	20/3	10	40/3	50/3	7	14	20/3	40/3	20
Λ_0	6	12	18	24	5	10	15	20	25	10	20	10	20	30

Table 2: Coefficients of the fundamental weights Λ_i with respect the α_i basis. Non affine matrix obtained from the elimination of the root with Coxeter label 1 from the vector $(1122)[6]$.

4.4 Example: the (2334) and the rest of SL Berger graphs

As before, one can construct the Berger matrix and graph based of the vector $\vec{k}_4 = (2334)[12]$ from CY_3 . The graph associated to this vector appears in Fig.(5,center). The

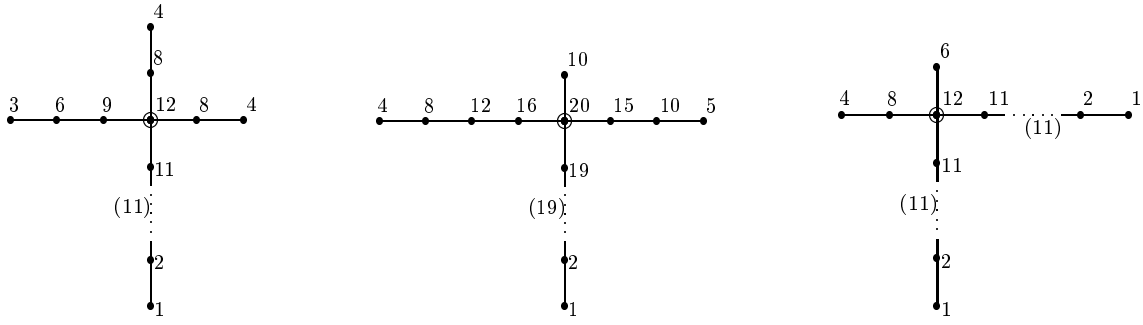


Figure 7: Berger Graphs corresponding to the vectors (from left to right) $(1344)[12]$, $(145, 10)[20]$ and $(1146)[12]$.

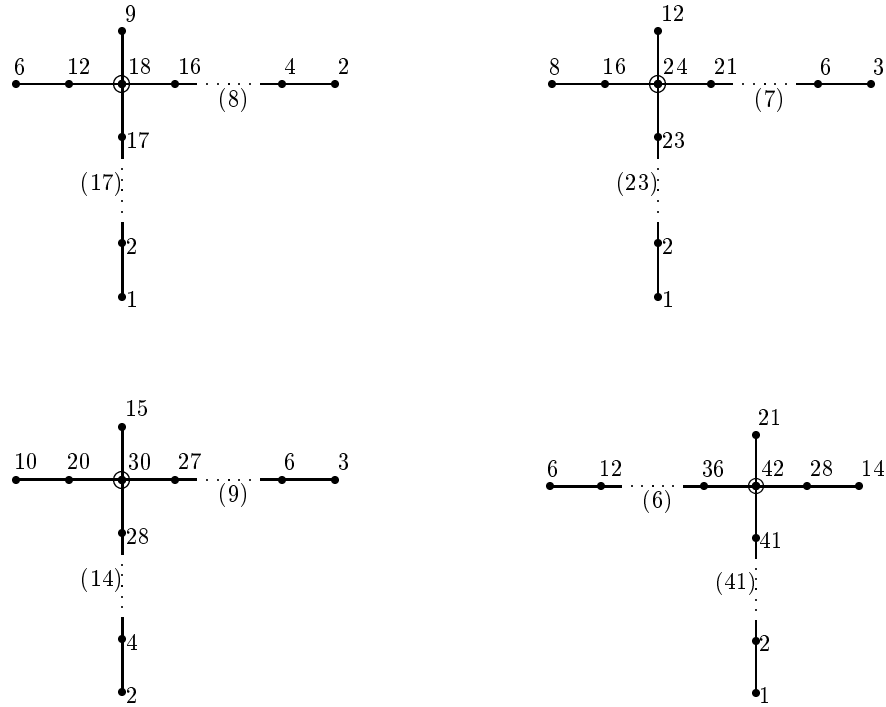


Figure 8: Berger Graphs corresponding to the vectors (from left to right) $(1269)[18]$, $(138, 12)[24]$, $(23, 10, 15)[30]$ and $(16, 14, 21)[42]$.

result is the following, rank = 14, 15×15 symmetric matrix

$$CY3B(2334) = \left(\begin{array}{cccccc|cccc|cccc|c} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 3 \end{array} \right) \quad (4.7)$$

A system of roots $(\alpha_i, i = 1, \dots, 15)$ in a orthonormal basis. Considering the orthonormal canonical basis $(\{e_i\}, i = 1, \dots, 14)$, we obtain:

$$\begin{aligned}
\alpha_{a5} &= e_{13} - e_{14} \\
&\dots \\
\alpha_{a1} &= e_9 - e_{10} \\
\alpha_{b3} &= e_7 + e_8 \\
\alpha_{b2} &= -1/2(e_2 - e_1 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8) \\
\alpha_{b1} &= e_2 - e_1 \\
\alpha_{c3} &= e_6 - e_5 \\
\alpha_{c2} &= e_5 - e_4 \\
\alpha_{c1} &= e_4 - e_3 \\
\alpha_{d2} &= -1/2(e_1 + e_2 - e_9 - e_{10} - e_{11} - e_{12} - e_{13} - e_{14}) \\
\alpha_{d1} &= e_1 + e_2 \\
\alpha_0 &= e_3 - e_2 - e_9
\end{aligned}$$

The null eigenvector has coordinates in the α basis, or Coxeter labels, $\mu = (4, 8, 3, 6, 9, 3, 6, 9, 2, 4, 6, 8, 10, 12)$. The Coxeter number is $h = 90$. For this affine matrix we can obtain *three* non-equivalent derived non affine, positive definite, matrices of dimension 14 simply by eliminating one of the columns and rows. They correspond to the elimination of any one of the extreme nodes labeled with Coxeter labels 2, 3, 4. The determinants are 8, 18, 32. The fundamental weights corresponding to the elimination of the Coxeter label 2, 3 and 4 nodes are given in tables (3, 4, 5) respectively.

The analysis of the rest of the graphs, matrices and obtention of roots and vectors is completely similar to the examples presented until now and offer no difficulty: all the information necessary to recover these cases have been already presented in table (1). The complete list of the graphs is explicitly presented in the Figs.(5, 6, 7, 8). Additional examples of root systems are presented in tables(7, 8) and those of weight vectors in table(6).

5. Summary, additional comments and conclusions

The interest to look for new algebras beyond Lie algebras started from the $SU(2)$ - conformal theories (see for example [19, 20]). One can think that geometrical concepts, in particular algebraic geometry, could be a natural and more promising way to do this. This marriage of algebra and geometry has been useful in both ways. Let us remind that to prove mirror symmetry of Calabi-Yau spaces, the greatest progress was reached with using the techniques of Newton reflexive polyhedra in Ref.[4].

In this work we have continued the study of the structure of graphs obtained from CY_3 reflexive polyhedra focusing on the description of fourteen “simply laced” cases, those graphs obtained from three dimensional spaces with K3 fibers which lead to symmetric matrices. We have studied both the affine and, derived from them, non-affine cases. We have presented root and weight structurea for them. We have studied in particular those

	α_{a4}	α_{a3}	α_{a2}	α_{a1}	α_{b3}	α_{b2}	α_{b1}	α_{c3}	α_{c2}	α_{c1}	α_{d2}	α_{d1}	α_0
Λ_{a4}	2	3	4	5	3/2	3	9/2	3/2	3	9/2	2	4	6
Λ_{a3}	3	6	8	10	3	6	9	3	6	9	4	8	12
Λ_{a2}	4	8	12	15	9/2	9	27/2	9/2	9	27/2	6	12	18
Λ_{a1}	5	10	15	20	6	12	18	6	12	18	8	16	24
Λ_{b3}	3/2	3	9/2	6	21/8	17/4	47/8	15/8	15/4	45/8	5/2	5	15/2
Λ_{b2}	3	6	9	12	17/4	17/2	47/4	15/4	15/2	45/4	5	10	15
Λ_{b1}	9/2	9	27/2	18	47/8	47/4	141/8	45/8	45/4	135/8	15/2	15	45/2
Λ_{c3}	3/2	3	9/2	6	15/8	15/4	45/8	21/8	17/4	47/8	5/2	5	15/2
Λ_{c2}	3	6	9	12	15/4	15/2	45/4	17/4	17/2	47/4	5	10	15
Λ_{c1}	9/2	9	27/2	18	45/8	45/4	135/8	47/8	47/4	141/8	15/2	15	45/2
Λ_{d2}	2	4	6	8	5/2	5	15/2	5/2	5	15/2	4	7	10
Λ_{d1}	4	8	12	16	5	10	15	5	10	15	7	14	20
Λ_0	6	12	18	24	15/2	15	45/2	15/2	15	45/2	10	20	30

Table 3: Coefficients of the fundamental weights Λ_i with respect the α_i basis. Non affine matrix obtained from the elimination of the root with Coxeter label 2 from the vector (2334)[12].

<i>F.W.</i>	α_{a5}	α_{a4}	α_{a3}	α_{a2}	α_{a1}	α_{b2}	α_{b1}	α_{c3}	α_{c2}	α_{c1}	α_{d2}	α_{d1}	α_0
Λ_{a5}	7/6	4/3	3/2	5/3	11/6	2/3	4/3	1/2	1	3/2	2/3	4/3	2
Λ_{a4}	4/3	8/3	3	10/3	11/3	4/3	8/3	1	2	3	4/3	8/3	4
Λ_{a3}	3/2	3	9/2	5	11/2	2	4	3/2	3	9/2	2	4	6
Λ_{a2}	5/3	10/3	5	20/3	22/3	8/3	16/3	2	4	6	8/3	16/3	8
Λ_{a1}	11/6	11/3	11/2	22/3	55/6	10/3	20/3	5/2	5	15/2	10/3	20/3	10
Λ_{b2}	2/3	4/3	2	8/3	10/3	2	3	1	2	3	4/3	8/3	4
Λ_{b1}	4/3	8/3	4	16/3	20/3	3	6	2	4	6	8/3	16/3	8
Λ_{c3}	1/2	1	3/2	2	5/2	1	2	3/2	2	5/2	1	2	3
Λ_{c2}	1	2	3	4	5	2	4	2	4	5	2	4	6
Λ_{c1}	3/2	3	9/2	6	15/2	3	6	5/2	5	15/2	3	6	9
Λ_{d2}	2/3	4/3	2	8/3	10/3	4/3	8/3	1	2	3	2	3	4
Λ_{d1}	4/3	8/3	4	16/3	20/3	8/3	16/3	2	4	6	3	6	8
Λ_0	2	4	6	8	10	4	8	3	6	9	4	8	12

Table 4: Coefficients of the fundamental weights Λ_i with respect the α_i basis. Non affine matrix obtained from the elimination of the root with Coxeter label 3 from the vector (2334)[12].

graphs leading to generalizations of the exceptional simply laced cases $E_{6,7,8}$ and $E_{6,7,8}^{(1)}$. The graphs and matrices of these simply laced graphs, both, those already known of dimension 1,2,3 and those new of dimension 4 share a number of simple characteristics. The cases of dimension 1,2,3 are well known and correspond to the classical Cartan Lie algebras. The main objective of this work has been to enlarge this list with graphs obtained by vectors of dimension four (corresponding to CY3). In dim=4, corresponding to K3-sliced CY_3 spaces, we have singled out by inspection the following 14 RW-reflexive vectors from the total of 95- K3-vectors (1111),(1122), (1113), (1124), (2334),(1344),(1236), (1225),(14510), (1146),(1269),(1,3,8,12),(2,3,10,15)(1,6,14,21). Coxeter numbers can be assigned in a con-

$F.W.$	α_{a5}	α_{a4}	α_{a3}	α_{a2}	α_{a1}	α_{b3}	α_{b2}	α_{b1}	α_{c3}	α_{c2}	α_{c1}	α_{d1}	α_0
Λ_{a5}	1	1	1	1	1	1/4	1/2	3/4	1/4	1/2	3/4	1/2	1
Λ_{a4}	1	2	2	2	2	1/2	1	3/2	1/2	1	3/2	1	2
Λ_{a3}	1	2	3	3	3	3/4	3/2	9/4	3/4	3/2	9/4	3/2	3
Λ_{a2}	1	2	3	4	4	1	2	3	1	2	3	2	4
Λ_{a1}	1	2	3	4	5	5/4	5/2	15/4	5/4	5/2	15/4	5/2	5
Λ_{b3}	1/4	1/2	3/4	1	5/4	9/8	5/4	11/8	3/8	3/4	9/8	3/4	3/2
Λ_{b2}	1/2	1	3/2	2	5/2	5/4	5/2	11/4	3/4	3/2	9/4	3/2	3
Λ_{b1}	3/4	3/2	9/4	3	15/4	11/8	11/4	33/8	9/8	9/4	27/8	9/4	9/2
Λ_{c3}	1/4	1/2	3/4	1	5/4	3/8	3/4	9/8	9/8	5/4	11/8	3/4	3/2
Λ_{c2}	1/2	1	3/2	2	5/2	3/4	3/2	9/4	5/4	5/2	11/4	3/2	3
Λ_{c1}	3/4	3/2	9/4	3	15/4	9/8	9/4	27/8	11/8	11/4	33/8	9/4	9/2
Λ_{d1}	1/2	1	3/2	2	5/2	3/4	3/2	9/4	3/4	3/2	9/4	2	3
Λ_0	1	2	3	4	5	3/2	3	9/2	3/2	3	9/2	3	6

Table 5: Coefficients of the fundamental weights Λ_i with respect the α_i basis. Non affine matrix obtained from the elimination of the root with Coxeter label 4 from the vector (2334)[12].

$F.W.$	α_{a4}	α_{a3}	α_{a2}	α_{a1}	α_{b5}	α_{b4}	α_{b3}	α_{b2}	α_{b1}	α_{c5}	α_{c4}	α_{c3}	α_{c2}	α_{c1}	α_{d1}	α_0
Λ_{a4}	2	3	4	5	1	2	3	4	5	1	2	3	4	5	3	6
Λ_{a3}	3	6	8	10	2	4	6	8	10	2	4	6	8	10	6	12
Λ_{a2}	4	8	12	15	3	6	9	12	15	3	6	9	12	15	9	18
Λ_{a1}	5	10	15	20	4	8	12	16	20	4	8	12	16	20	12	24
Λ_{b5}	1	2	3	4	5/3	7/3	3	11/3	13/3	5/6	5/3	5/2	10/3	25/6	5/2	5
Λ_{b4}	2	4	6	8	7/3	14/3	6	22/3	26/3	5/3	10/3	5	20/3	25/3	5	10
Λ_{b3}	3	6	9	12	3	6	9	11	13	5/2	5	15/2	10	25/2	15/2	15
Λ_{b2}	4	8	12	16	11/3	22/3	11	44/3	52/3	10/3	20/3	10	40/3	50/3	10	20
Λ_{b1}	5	10	15	20	13/3	26/3	13	52/3	65/3	25/6	25/3	25/2	50/3	125/6	25/2	25
Λ_{c5}	1	2	3	4	5/6	5/3	5/2	10/3	25/6	5/3	7/3	3	11/3	13/3	5/2	5
Λ_{c4}	2	4	6	8	5/3	10/3	5	20/3	25/3	7/3	14/3	6	22/3	26/3	5	10
Λ_{c3}	3	6	9	12	5/2	5	15/2	10	25/2	3	6	9	11	13	15/2	15
Λ_{c24}	8	12	16	10/3	20/3	10	40/3	50/3	11/3	22/3	11	44/3	52/3	10	20	
Λ_{c1}	5	10	15	20	25/6	25/3	25/2	50/3	125/6	13/3	26/3	13	52/3	65/3	25/2	25
Λ_{d1}	3	6	9	12	5/2	5	15/2	10	25/2	5/2	5	15/2	10	25/2	8	15
Λ_0	6	12	18	24	5	10	15	20	25	5	10	15	20	25	15	30

Table 6: Coefficients of the fundamental weights Λ_i with respect the α_i basis. Non affine matrix obtained from the elimination of one of the root from the vector (1113)[6].

sistent way both geometrically and algebraically. Genuine Berger matrices are assigned to them with specially simple properties: they are symmetric and affine. In addition, each of these graphs and matrices seems not be “extendable”: in contradistinction to other cases [16], no other graphs and Berger matrices can be obtained from them simply adding more nodes to any of the legs. In this sense, these graphs are “exceptional”. As with the classical exceptional graphs, series can be traced among them. Apparently these fourteen vectors

$\begin{aligned} \alpha_{a11} &= e_{19} - e_{20} \\ &\dots \\ \alpha_{a2} &= e_{10} - e_{11} \\ \alpha_{a1} &= e_9 - e_{10} \\ \alpha_{b5} &= -1/2(e_7 - e_6 - e_5 - e_4 - e_3 - e_2 - e_1 - e_8) \\ \alpha_{b4} &= e_7 - e_6 \\ &\dots \\ \alpha_{b1} &= e_4 - e_3 \\ \alpha_{c3} &= 1/3(2e_8 - e_1 - e_2 + e_9 + e_{10} + \dots + e_{20}) \\ \alpha_{c2} &= e_1 - e_8 \\ \alpha_{c1} &= e_2 - e_1 \\ \alpha_{d1} &= 1/2(e_2 + e_1 + e_8 - e_3 - e_4 - e_5 - e_6 - e_7) \\ \alpha_0 &= e_3 - e_2 - e_9 \end{aligned}$	$\begin{aligned} \alpha_{a11} &= e_{17} - e_{18} \\ &\dots \\ \alpha_{a1} &= e_7 - e_8 \\ \alpha_{b3} &= 1/3(2e_5 - e_1 + e_7 + e_8 + \dots + e_{18}) \\ \alpha_{b2} &= e_1 - e_5 \\ \alpha_{b1} &= e_2 - e_1 \\ \alpha_{c2} &= -1/2(e_4 - e_3 - e_2 - e_1 - e_5 + pe_6) \\ \alpha_{c1} &= e_4 - e_3 \\ \alpha_{d2} &= -1/2(e_4 + e_3 + e_2 + e_1 + e_5 + pe_6) \\ \alpha_{d1} &= e_4 + e_3 \\ \alpha_0 &= e_3 - e_2 - e_7 \end{aligned}$
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Table 7: Roots for the Berger cases (Left) (1236)[12] and (Right) (1344)[12] where $p = \sqrt{3}$.

$\begin{aligned} \alpha_{a11} &= 1/11(11e_1 - 10e_2 + e_3 + e_4 + \dots + e_{24}) \\ \alpha_{a10} &= e_2 - e_3 \\ &\dots \\ \alpha_{a1} &= e_{11} - e_{12} \\ \alpha_{b11} &= 1/11(-11e_{25} + 10e_{24} - e_{23} - e_{22} - \dots - e_2) \\ \alpha_{b10} &= e_{23} - e_{24} \\ &\dots \\ \alpha_{b1} &= e_{14} - e_{15} \\ \alpha_{c2} &= 1/2(e_{13} + e_{26}) - 1/4(e_{12} + e_{11} + \dots + e_1) + \\ &\quad + 1/4(e_{14} + e_{15} + \dots + e_{25}) \\ \alpha_{c1} &= 1/2(e_{13} + e_{26}) + 1/4(e_{12} + e_{11} + \dots + e_1) - \\ &\quad - 1/4(e_{14} + e_{15} + \dots + e_{25}) \\ \alpha_{d1} &= e_{13} - e_{26} \\ \alpha_0 &= e_{12} - e_{13} - e_{14} \end{aligned}$	$\begin{aligned} \alpha_{a14} &= e_{22} - e_{23} \\ &\dots \\ \alpha_{a1} &= e_9 - e_{10} \\ \alpha_{b9} &= -1/2(e_4 - e_3) + 1/4(e_2 + e_1 + \dots + e_8) + \\ &\quad + 1/4(e_9 + \dots + e_{23}) \\ \alpha_{b8} &= e_7 - e_8 \\ &\dots \\ \alpha_{b1} &= e_2 - e_1 \\ \alpha_{c2} &= 1/6(7e_8 + e_7 + \dots + e_1 + e_2 - e_9 - e_{10} - e_{23}) \\ \alpha_{c1} &= e_4 - e_3 \\ \alpha_{d1} &= e_4 + e_3 \\ \alpha_0 &= e_3 - e_2 - e_9 \end{aligned}$
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Table 8: Roots for the Berger cases (Left) (1146)[12] and (Right) (2, 3, 10, 15)[30].

are the only ones from the the total of 95 vectors which lead to this kind of symmetric matrices.

It is very well known, by the Serre theorem, that Dynkin diagrams defines one-to-one Cartan matrices and these ones Lie or Kac-Moody algebras. In this work, we have generalized some of the properties of Cartan matrices for Cartan-Lie and Kac-Moody algebras into a new class of affine, and non-affine Berger matrices. We arrive then to the obvious conclusion that any algebraic structure emerging from this can not be a CLA or KMA algebra. The main difference of these matrices with respect previous definitions being in the values that diagonal elements of the matrices can take. In Calabi-Yau CY3 spaces, new entries with norm equal to 3 are allowed. The choice of this number can be related to two

$$\begin{aligned}
\alpha_{a41} &= e_{49} - e_{50} \\
&\dots \\
\alpha_{a1} &= e_9 - e_{10} \\
\alpha_{b6} &= 1/6(5e_5 - e_4 - e_3 \dots - e_2 - e_1 + e_9 + e_{10} + \dots + e_{50}) \\
\alpha_{b5} &= e_1 - e_5 \\
\alpha_{b4} &= e_1 - e_5 \\
\alpha_{b3} &= 1/3(2e_5 - e_1 + e_7 + e_8 + \dots + e_{18}) \\
\alpha_{b2} &= e_1 - e_5 \\
\alpha_{b1} &= e_2 - e_1 \\
\alpha_{c2} &= -1/4(5e_4 + e_3 + e_2 + e_1 + e_2 + \dots + e_5) \\
\alpha_{c1} &= e_4 - e_3 \\
\alpha_{d1} &= -1/2(e_4 + e_3 - e_2 - e_1 - \dots - e_5) \\
\alpha_0 &= e_3 - e_2 - e_9
\end{aligned}$$

Table 9: Roots for the $(1, 6, 14, 21)$ [30] Berger case.

facts: First, we should take in mind that in higher dimensional Calabi-Yau spaces resolution of singularities should be accomplished by more topologically complicated projective spaces: while for resolution of quotient singularities in K3 case one should use the CP^1 with Euler number 2, the, Euler number 3, CP^2 space could be used for the resolution of singularities in CY_3 space in a non-irreducible way. The second fact is related to the cubic matrix theory[21], where a ternary operation is defined and in which the S_3 group naturally appears. One conjecture, draft from the fact of the underlying UCYA construction, is that, as Lie and affine Kac-Moody algebras are based on a binary composition law; the emerging picture from the consideration of these graphs could lead us to algebras including simultaneously different n-ary composition rules. Of course, the underlying UCYA construction could manifest in other ways: for example in giving a framework for a higher level linking of algebraic structures: Kac-Moody algebras among themselves and with any other hypothetical algebra generalizing them. Thus, putting together UCYA theory and graphs from reflexive polyhedra, we expect that iterative application of non-associative n-ary operations give us not only a complete picture of the RWV, but allow us in addition to establish “dynamical” links among RWV vector and graphs of different dimensions and, in a further step, links between singularity blow-up and possibly new generalized physical symmetries.

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